Find the general solution of $x^2y'' - xy' + 5y = x\csc(2\ln x)$. $g = \frac{x\csc(2\ln x)}{x^2} = \frac{1}{x\sin(2\ln x)}$ **SCORE: _____ / 30 PTS**

$r^2 - 2r + 5 = 0$

$r = 1 \pm 2i$

$y_h = C_1 x\cos(2\ln x) + C_2 x\sin\ln x$

2 EACH EXCEPT AS NOTED

$$W = \begin{vmatrix} x\cos(2\ln x) & x\sin(2\ln x) \\ \cos(2\ln x) - 2\sin(2\ln x) & \sin(2\ln x) + 2\cos(2\ln x) \end{vmatrix} = 2x \quad 4$$

$$y_p = -x\cos(2\ln x)\int \frac{1}{\cancel{x\sin(2\ln x)}}\,\frac{\cancel{x\sin(2\ln x)}}{2x}\,dx \quad 4$$

$$+\, x\sin(2\ln x)\int \frac{1}{\cancel{x}\sin(2\ln x)}\,\frac{\cancel{x}\cos(2\ln x)}{2x}\,dx \quad 4 \leftarrow u = \sin(2\ln x), \; du = \frac{2\cos(2\ln x)}{x}dx$$

$$= -x\cos(2\ln x)\cdot\frac{1}{2}\ln|x| + x\sin(2\ln x)\cdot\frac{1}{4}\ln|\sin(2\ln x)| \quad 4$$

$$= -\frac{1}{2}x\ln|x|\cos(2\ln x) + \frac{1}{4}\ln|\sin(2\ln x)|\sin(2\ln x)$$

$$y = -\frac{1}{2}x\ln|x|\cos(2\ln x) + \frac{1}{4}\ln|\sin(2\ln x)|\sin(2\ln x) + C_1 x\cos(2\ln x) + C_2 x\sin(2\ln x)$$

Find the general solution of $y''' - y'' + 4y' - 4y = 20\cos 2t$. (2) EACH EXCEPT AS NOTED **SCORE: _____ / 40 PTS**

$$r^3 - r^2 + 4r - 4 = 0 \qquad r = \pm 2i, 1$$

$$r^2(r-1) + 4(r-1) = 0 \longrightarrow y_h = c_1\cos 2t + c_2 \sin 2t + c_3 e^t$$

$$(r^2+4)(r-1) = 0$$

$$y_p = At\cos 2t + Bt\sin 2t \quad (4)$$

$$y_p' = A\cos 2t - 2At\sin 2t + 2Bt\cos 2t + B\sin 2t = (2Bt + A)\cos 2t + (-2At + B)\sin 2t \quad (4)$$

$$y_p'' = 2B\cos 2t + (-4Bt - 2A)\sin 2t + (-4At + 2B)\cos 2t - 2A\sin 2t = (-4At + 4B)\cos 2t + (-4Bt - 4A)\sin 2t \quad (4)$$

$$y_p''' = -4A\cos 2t + (8At - 8B)\sin 2t + (-8Bt - 8A)\cos 2t - 4B\sin 2t$$

$$= (-8Bt - 12A)\cos 2t + (8At - 12B)\sin 2t \quad (4)$$

$$-y_p'' \quad + (4At - 4B)\cos 2t + (4Bt + 4A)\sin 2t$$

$$+4y_p' \quad + (8Bt + 4A)\cos 2t + (-8At + 4B)\sin 2t \quad (4)$$

$$-4y_p \quad -4At\cos 2t - 4Bt\sin 2t$$

$$= (-8A - 4B)\cos 2t + (4A - 8B)\sin 2t = 20\cos 2t \quad (4)$$

$$\left.\begin{aligned} -8A - 4B &= 20 \\ 4A - 8B &= 0 \end{aligned}\right| \longrightarrow \begin{aligned} 2A + B &= -5 \\ A - 2B &= 0 \end{aligned} \longrightarrow \left.\begin{aligned} A &= -2 \\ B &= -1 \end{aligned}\right|$$

$$y = -2t\cos 2t - t\sin 2t + c_1\cos 2t + c_2\sin 2t + c_3 e^t$$

Use elimination (as shown in lecture) to solve the system

2 EACH EXCEPT AS NOTED

SCORE: _____ / 45 PTS

$$(3D+2)[x] + 2D[y] = 2t+1$$
$$(D+6)[x] + (D+2)[y] = 2t+1$$

$$(D+2)(3D+2)[x] + (D+2)(2D)[y] = (D+2)[2t+1] = 2+4t+2 = \underline{4t+4}$$
$$2D(D+6)[x] + 2D(D+2)[y] = 2D[2t+1] = \underline{4} \ 1$$

4 $\underline{(D^2-4D+4)[x] = 4t}$

$r = 2, 2$

$\underline{x_h = c_1e^{2t} + c_2te^{2t}}$

$\underline{x_p = At+B}$

$\underline{x_p' = A}$ 1

$\underline{x_p'' = 0}$ 1

$$x_p'' - 4x_p' + 4x_p = -4A + 4At + 4B = \underline{4At + (-4A+4B)} = 4t$$

$4A = 4 \rightarrow A = 1$

$-4A+4B = 0 \rightarrow B = A = 1$

$x = \underline{t + 1 + c_1e^{2t} + c_2te^{2t}}$

$$(D+6)(3D+2)[x] + (D+6)(2D)[y] = (D+6)[2t+1] = 2+12t+6 = \underline{12t+8}$$
$$(3D+2)(D+6)[x] + (3D+2)(D+2)[y] = (3D+2)[2t+1] = 6+4t+2 = \underline{4t+8}$$

$\underline{(D^2-4D+4)[y] = -8t}$

$y_h = k_1e^{2t} + k_2te^{2t}$

$y_p = Ct+E$

$$y_p'' - 4y_p' + 4y_p = 4Ct + (-4C+4E) = -8t$$

$4C = -8 \rightarrow C = -2$

$-4C+4E = 0 \rightarrow E = C = -2$

1 1

$y = \underline{-2t - 2 + k_1e^{2t} + k_2te^{2t}}$

$$(3D+2)[x] + 2D[y] = 3 + 6c_1e^{2t} + 6c_2te^{2t}$$
$$+ 3c_2e^{2t}$$
$$+ 2t + 2 + 2c_1e^{2t} + 2c_2te^{2t}$$
$$- 4 + 4k_1e^{2t} + 4k_2te^{2t}$$
$$+ 2k_2e^{2t}$$

4

4

$$= \underline{2t + 1 + (8c_1 + 3c_2 + 4k_1 + 2k_2)e^{2t} + (8c_2 + 4k_2)te^{2t}}$$

$8c_2 + 4k_2 = 0 \rightarrow \underline{k_2 = -2c_2}$

$8c_1 + 3c_2 + 4k_1 + 2k_2 = 0$

$8c_1 - c_2 + 4k_1 = 0 \rightarrow \underline{k_1 = -2c_1 + \frac{1}{4}c_2}$

$x = t + 1 + c_1e^{2t} + c_2te^{2t}$

$\underline{y = -2t - 2 + (-2c_1 + \frac{1}{4}c_2)e^{2t} - 2c_2te^{2t}}$

$y = e^{-x}$ is a solution of $xy'' + (2x+1)y' + (x+1)y = 0$. **SCORE: _____ / 35 PTS**

$y = 3x^2 - 4x$ is a particular solution of $xy'' + (2x+1)y' + (x+1)y = 3x^3 + 11x^2 - 4$.

Solve the initial value problem $xy'' + (2x+1)y' + (x+1)y = 8 - 22x^2 - 6x^3, \quad y(-1) = 0, \quad y'(-1) = 0$.

$$= -2(3x^3 + 11x^2 - 4)$$

$$\text{so } y_p = -2(3x^2 - 4x) = 8x - 6x^2$$

$$y_2 = ve^{-x}$$

$$y_2' = v'e^{-x} - ve^{-x}$$

$$y_2'' = v''e^{-x} - 2v'e^{-x} + ve^{-x}$$

(2) EACH EXCEPT AS NOTED

$$xy_2'' + (2x+1)y_2' + (x+1)y_2$$

$$= e^{-x}[xv'' - 2xv' + xv + (2x+1)v' - (2x+1)v + (x+1)v]$$

$$= e^{-x}(xv'' + v') = 0$$

LET $u = v'$ $\quad x\frac{du}{dx} + u = 0$

$$\frac{1}{u}du = -\frac{1}{x}dx$$

$$\ln|u| = -\ln|x| \quad (1)$$

$$u = x^{-1}$$

$$v' = x^{-1}$$

$$v = \ln|x|$$

$$y_2 = e^{-x}\ln|x|$$

$$y = 8x - 6x^2 + c_1e^{-x} + c_2e^{-x}\ln|x|$$

$$y' = 8 - 12x - c_1e^{-x} - c_2e^{-x}\ln|x| + c_2x^{-1}e^{-x}$$

$$y(-1) = -14 + c_1e = 0$$

$$y'(-1) = 20 - c_1e - c_2e = 0$$

$$c_1 = \frac{14}{e}$$

$$20 - 14 - c_2e = 0 \rightarrow c_2 = \frac{6}{e}$$

$$y = 8x - 6x^2 + 14e^{-x-1} + 6e^{-x-1}\ln|x|$$